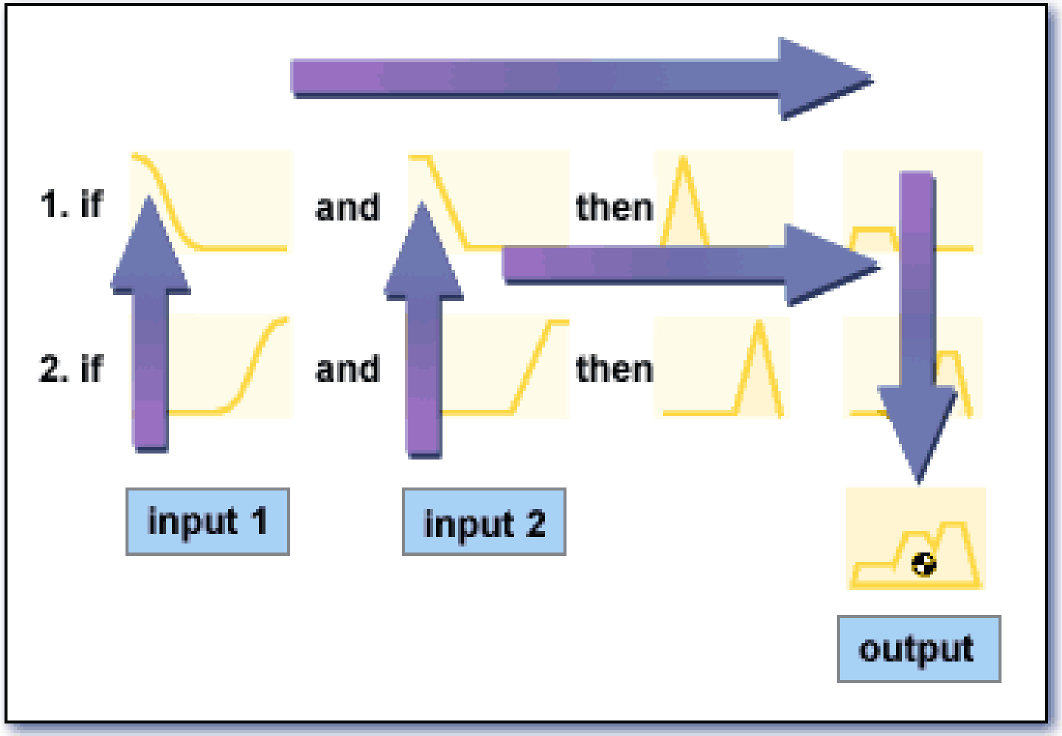
AS PER THE SYLLABUS OF BPUT FOR SEVENTH SEMESTER OF AE&IE BRANCH.

RIT,
BERHAMPUR

SOFT COMPUTING (PECS 3401)-FUZZY LOGIC



Interpreting the Fuzzy Inference Diagram

CHAPTER 01
INTRODUCTION TO SOFT COMPUTING

ORIGIN OF SOFT COMPUTING

Artificial intelligence is a branch of computer science dealing with building of system that exhibits *automation in intelligent behaviour*. But *intelligence* is not very well defined due to which the tasks associated with intelligence i.e. *learning, intuition, creativity* and *decision making* also seems to partially understood.

This quest of building intelligent systems gave rise to new techniques that aided a lot in building intelligent problem solvers. Some of them are *expert system, neural networks, fuzzy logic, cellular automata* and *probabilistic reasoning*. Out of this *fuzzy logic, neural networks* and *probabilistic reasoning* are called as *soft computing*. The term soft computing was coined by Lotfi A. Zadeh.

Soft computing differs from *hard computing* (conventional computing) in its *tolerance* to *imprecise, uncertain* and *partial truth*. Hard computing basically deals with mathematical approaches that demands a great degree of precision and accuracy. But in engineering problems, it is very difficult to determine the input with great degree of precision. Hence a best estimate is made to find the solution which restricts the use of mathematical approaches, especially in inverse problems.

The application of soft computing to the inverse problems is to exploit the *tolerance for imprecision, uncertainty and partial truth to achieve tractability, robustness and low cost solutions.*

CONSTITUENTS OF SOFT COMPUTING

The basic components of soft computing are as follows:

- Fuzzy Systems (models uncertainty in the system)
- Neural Networks (models biological neuron of human brain)
- Genetic Algorithm (selection of a good solution from a solution set)

FUZZY LOGIC

It is a mathematical tool to model *uncertainty* in the system. This is applicable where *imperfection* in data is present. Basically used to represent *uncertainty* that arises due to *generality, vagueness, ambiguity, chance or incomplete knowledge*.

In *fuzzy sets* the members of the sets are associated with a value representing its *grade of membership* in the fuzzy set. This is in contrast to that of classical set called as *crisp set* i.e. the members are either belongs to the set or not. But each member in the fuzzy set is associated with a membership value between [0, 1].

Let A={ram, kiran, john} and their corresponding height are 6, 5.5 and 5.1 feets respectively. Now Height >6ft(90%), Height<5ft(10%), and Height=5ft(50%). So set A can be represented as {0.9/ram,0.5/kiran, 0.1/john} where the fractional values represents the membership values.

ARTIFICIAL NEURAL NETWORKS

ANN is based on learning and testing theory. A set of data is needed to learn things.
$f(x)=x^2$ in the range (0,7)

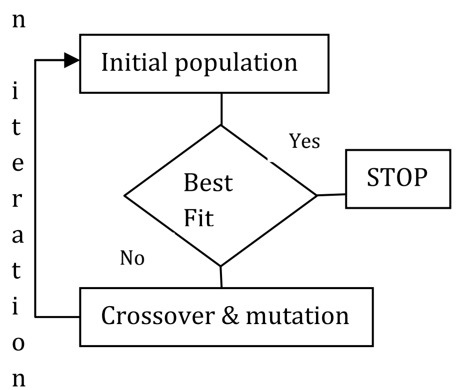
x	:	0	1	2	3	4	5	6	7
f(x)	:	0	1	4	9	16	25	36	49

There are massively highly interconnected N/W of processing elements called *neurons*. There are unorthodox search and optimization algorithms insipid by the biological process.

EVOLUTIONARY TECHNIQUES

This is also called as genetic process. It is used to mimic natural evolution. GA makes a random search through a given set of alternate solution to find the best alternate solution w.r.t the given criteria of goodness. e.g. ant colony optimization, swarm intelligence etc.

Some of the techniques that are combination of these basic components of soft compounding are called as *hybrid techniques*. Some of these are as follows: *neuro-fuzzy*, *neuro-GA*, *fuzzy-GA*.



ADVANTAGES OF SOFT COMPUTING

- Model based on human reasoning
- Models can be simple and accurate
- Fast computing
- Good in practice

APPLICATION AREAS

- Process control
- Robotics
- Optimization techniques
- Pattern recognition
- Time series forecasting
- Medicine and diagnosis

CHAPTER 02
FUZZY LOGIC

INTRODUCTION

Fuzzy logic is a tool which provides an inference morphology that enables appropriate human reasoning capabilities to be applied to knowledge based systems. The theory of fuzzy logic provided mathematical strength to capture the uncertainties associated with human cognitive processed such as thinking and reasoning.

ADVANTAGES OF FUZZY LOGIC

- There are an efficient tool for embedding human knowledge into useful algorithms.
- This can approximate any multi valued non linear function.
- These are applicable when mathematical models are unknown or impossible to obtain.
- Operates successfully under a lack of precise information.
- Are also appropriate tools in generic decision making processes.

DISADVANTAGES

- Human solution to the problem must exist and this knowledge must be structured.
- Number of rules increases exponentially with increases in number of inputs and number of fuzzy subsets per input variable.

PROBABILITY VS FUZZY

Fuzzy logic and probability theories are the most powerful tools to overcome the imperfections. Fuzzy logic mainly responsible for representing and processing of vague data. Probability theory is mainly responsible for representing and processing of uncertainty.

Probability measure

- Calculates the probability that an unknown variable 'x' ranging on 'u' hits the well know set 'A'.
- Before the event happens
- Measure theory
- Domain is 2D(Y/N).

Membership measure

- Calculates the membership of a well known variable 'x' ranging on 'u' hits the unknown set 'A'.
- After the event happens
- It will use the set theory
- Domain is 0-1.

Imperfection

Probability Theory

- Randomness
- Probability rules
- Before the event happens

Fuzzy logic

- Vagueness (ill defined)
- Fuzzy rules
- After the event happens

EXAMPLES OF PROBABILITY THEORY

- It is probable that it will rain a lot tomorrow.
- It is probable that the image will be very dark.
- It is probable that her new friend is handsome.

Name	Height	Membership value
Sohan	5'.2"	0.2
John	6'.1"	1
Mohan	4'.7"	0
Abraham	5'.8"	0.8

CRISP SET VS FUZZY SETS

Example 1:

Tall={Sohan, John, Mohan, Abraham} ----------------------------------> Crisp set.

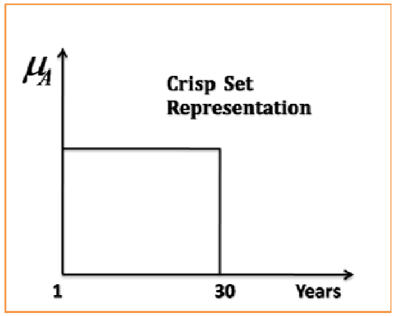
$$\mu_{Tall} = \begin{cases} 1, for\ x\ \in Tall \\ 0, for\ x \notin Tall \end{cases}$$ i.e. the value is either 0 or 1.

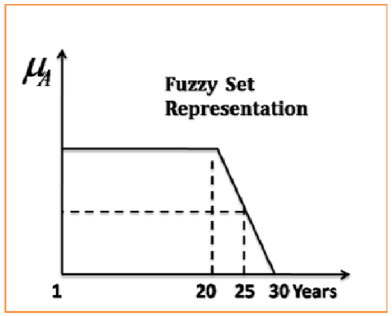
Tall={(sohan, 0.2), (john, 1), (Mohan, 0), (Abraham, 0.8)} ---------> Fuzzy set.

$\mu_{Tall} = \{[0-1] for\ all\ x\ \in Tall$ i.e. a number of membership functions ranging from 0 to 1.

Example 2:

A={x|x <= 30} where A-> youngness of people.





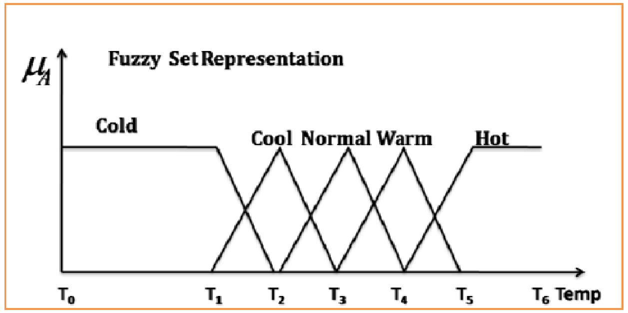
A set is a well defined collection of objects. Well defined means the objects either belongs to or does not belongs to the set. This is known as crispness and the sets are known as *crisp sets*.
A *fuzzy set* $\tilde{A}$ in a universe of discourse U is characterised by membership function μ_A which takes values in the interval 0 to 1 i.e. μ_A:U->[0,1].

If U contains finite numbers of elements, let it contain n elements, fuzzy set $\tilde{A}$ can be denoted by: $\tilde{A}$={ $\mu_A(u_1)/u_1$, $\mu_A(u_2)/u_2$, $\mu_A(u_3)/u_3$, ... , $\mu_A(u_n)/u_n$}.

Example 3:
X={1,2,3,4,5} and its Comfortness={(1,0.1), (2,0.4), (3,0.8), (4,1), (5,0.7)}.
The elements of the above fuzzy set are crisp sets.

Example 4:

Temperature:	Sensing:
T0-T2	Cold
T1-T3	Cool
T2-T4	Normal
T3-T5	Warm
T4-T6	Hot



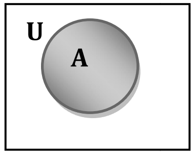
As we discussed earlier Tall={(sohan, 0.2), (john, 1), (mohan,0), (Abraham, 0.8)}. and similarly the opposite of this can be found by subtracting the membership values form 1. Short={(sohan, 0.8), (john, 0), (mohan,1), (Abraham, 0.2)}. ($\tilde{A}'$=1-$\tilde{A}$)

CHARACTERISTIC CRISP SET

Set: Set is a collection of well defined objects. A={1, 2, 3, ...}.

Universe of discourse: It is set that contain all possible objects from which other set can be formed. Universal set of all numbers is a Euclidean Space.

Venn diagram: It is a pictorial representation of set. The venn diagram for the set A and its universal set U is represented as follows.



Membership: If an element is x is a member of set A then it is represented by the symbol ∈ read as "belongs to". And when x is not a member of A then it is represented by the symbol ∉ read as "not belongs to". ($x \in A$).

Cardinality: The number of elements in a set is called as cardinality of the set. It is represented as |A| or #A. e.g. A={1,2,3,4} then |A|=4.

Family of sets: A sets whose members are sets themselves are called as family of sets. e.g. A={{1,2,3},{2,3,4,5},{1,3,5}}.

Null/empty sets: A set containing no elements is called a null set or empty set. It is represented as ∅ or {}. And |∅|=0 as it contain no elements.

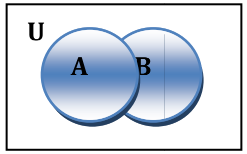
Singleton set: A set with a single element is called as singleton set. It has cardinality one. e.g. A={a} is a singleton set.

Subset: Given two set A and B over the universal set U. If all the elements of B is contained in the set A, then B is said to be a subset of A. It is represented as $B \subset A$.

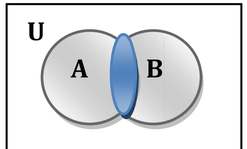
Superset: Given two sets A and B over the universal set U. If all the elements of B is contained in the set A, then A is said to be a superset of B. It is represented as $A \supset B$.

Power set: A power set is a collection of all subsets of a given sets including null set. For a set A={1,2}, the power set is2^A={{1}, {2}, {1, 2}, {}}.

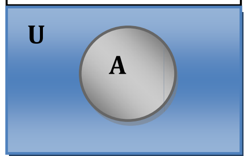
OPERATIONS ON CRISP SETS



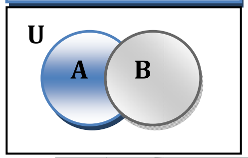
Union: Union operation on two sets A and B is another set represented by AUB that contains all the elements that are present in A and B. e.g. A={a,b,c} and B={b,c,d,f}, then AUB={a,b,c,d,f}.



Intersection: Intersection operation on two sets A and B is another set represented by AnB that contains elements that are present in A as well as B. e.g. A={a,b,c} and B={b,c,d,f}, then AnB={b,c}.

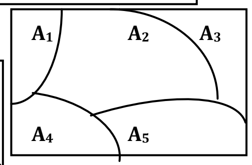


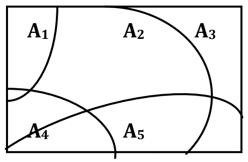
Complement: Complentation operation on a set A of a universal set U is a set represented by A^C that contain elements that are there in U but not in A.



Difference: Difference operation on two set A and B is another set represented by A-B that contain elements that are only in A but not in B.

Partition: Division of set into many subsets such that the union of all the subsets gives back the same set.





Covering: Division of set into many subset such that they have common element.

Rule of Addition: Given a partition on A where A_i, i=1,2,...,n are its non-empty subsets then, $|A| = |\bigcup_{i=0}^{n} A_i| = \sum_{i=1}^{n} |A_i|$.

Rules of Inclusion and Exclusion: Rules of addition cannot be applied to covering of a set A. In this case we use principle of inclusion and exclusion which is as follows, $|A| = \left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{i=1}^{n}\sum_{j=1}^{n}\left|A_i \cap A_j\right| + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left|A_i \cap A_j \cap\right.$ *Ak…−1n+1i=1nAi|*.

Problems 1:
Assume that |E|=600, |A|=300, |B|=225, |C|=160. Where A is the set of male students, B is the set of bowlers and C is the set of batsmen. Also given AnB be 100, 25 of whom are batsmen too i.e. AnBnC. And the total number of male batesmen i.e. AnC is 85. Determine the number of students who are i. Females, ii. Not Bowlers, iii. Not Batsmen and iv. female students who can bowl.

Solution: i. No. of females = |E|-|A|=600-300=300.
ii. Not Bowlers= |E|-|B|=600-225=375.
iii. Not Batsmen=|E|-|C|=600-160=440.
iv. Female who can bowl=|A'|n|B|=225-100=125.

Problem 2:
Given |E|=100, where E indicates a set of students who have chosen subjects form different streams in the computer science discipline, it is found that 32 study subjects chosen from CN stream, 20 from MMT stream, 45 from the SS stream. Also 15 study subjects CN and SS streams, 10 study subjects CN and MMT and 7 from both MMT and SS streams, and 30 do not study any subjects chosen from either of the three streams.

Solution:
|A' n B' n C'|=30. Given.
Then |A U B U C|'=30 by De Morgan's Law.
So we have |AUBUC|=|E|-|AUBUC|'
=100-30=70.
From the Principle of inclusion and exclusion
|AUBUC|=|A|+|B|+|C|-|AnB|-|BnC|-|CnA|+|AnBnC|
=>|AnBnC|=-|A|-|B|-|C|+|AnB|+|BnC|+|CnA|+|AUBUC|
-32-20-45+15+7+10+70= -97+102=5.

PROPERTIES OF FUZZY SETS

Commutative	$A \cup B = B \cup A$ *and* $A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$
Distributed	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotent	$A \cup A = A$ *and* $A \cap A = A$
Identity	$A \cup \emptyset = A;\ A \cup X = X;\ A \cap \emptyset = \emptyset;\ A \cap E = A$
Low of Absorption	$A \cup (A \cup B) = A$ *and* $A \cap (A \cup B) = A$
De Morgan's Law	$\overline{A \cup B} = \bar{A} \cap \bar{B}$ *and* $\overline{A \cap B} = \bar{A} \cup \bar{B}$

OPERATIONS ON FUZZY SETS

The set operations on fuzzy sets are similar to that of crisp sets. Whenever set operations are applied on one or more fuzzy sets it always results in a fuzzy sets. Let $A = \left\{\frac{0.2}{x_1}, \frac{0.3}{x_2}, \frac{0.7}{x_3}, \frac{0.5}{x_4}\right\}$, $B = \left\{\frac{0.6}{x_1}, \frac{0.1}{x_2}, \frac{0.6}{x_3}, \frac{0.4}{x_4}\right\}$ and $C = \left\{\frac{0.2}{x_1}, \frac{0.3}{x_2}, \frac{0.7}{x_3}, \frac{0.5}{x_4}\right\}$. Then the set operation are as follows:

Union: Let A and B be two fuzzy sets then the union operation results in a fuzzy set whose membership function is defined as follows $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$.
$\mu_{A\cup B}(x_1) = \max\{0.2, 0.6\} = 0.6$,
$\mu_{A\cup B}(x_2) = \max\{0.3, 0.1\} = 0.3$,
$\mu_{A\cup B}(x_3) = \max\{0.7, 0.6\} = 0.7$,
$\mu_{A\cup B}(x_4) = \max\{0.5, 0.4\} = 0.5$,
$A \cup B = \left\{\frac{0.6}{x_1}, \frac{0.3}{x_2}, \frac{0.7}{x_3}, \frac{0.5}{x_4}\right\}$

Intersection: Let A and B be two fuzzy sets then the intersection operation results in a fuzzy set whose membership function is defined as follows $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$.
$\mu_{A\cap B}(x_1) = \min\{0.2, 0.6\} = 0.2$,
$\mu_{A\cap B}(x_2) = \min\{0.3, 0.1\} = 0.1$,
$\mu_{A\cap B}(x_3) = \min\{0.7, 0.6\} = 0.6$,
$\mu_{A\cap B}(x_4) = \min\{0.5, 0.4\} = 0.4$,
$A \cap B = \left\{\frac{0.2}{x_1}, \frac{0.1}{x_2}, \frac{0.6}{x_3}, \frac{0.4}{x_4}\right\}$

Complementation: Let A be a fuzzy sets then the complementation operation results in a fuzzy set whose membership function is defined as follows $\mu_{A^c}(x) = \{1 - \mu_A(x)\}$.
$\mu_{A^c}(x_1) = \{1 - 0.2\} = 0.8$,
$\mu_{A^c}(x_2) = \{1 - 0.3\} = 0.7$,
$\mu_{A^c}(x_3) = \{1 - 0.7\} = 0.3$,

$\mu_{A^c}(x_4) = \{1 - 0.5\} = 0.5,$

$A^c = \left\{\frac{0.8}{x_1}, \frac{0.7}{x_2}, \frac{0.3}{x_3}, \frac{0.5}{x_4}\right\}$

Equality: Let A and B be two fuzzy sets then they are said to be equal if for every corresponding member of the two sets $\mu_A(x) = \mu_B(x)$.

$\mu_A(x_1) = \mu_C(x_1) = 0.2,$

$\mu_A(x_2) = \mu_C(x_2) = 0.3,$

$\mu_A(x_3) = \mu_C(x_3) = 0.7,$

$\mu_A(x_4) = \mu_C(x_4) = 0.5,$ Hence the fuzzy sets A and C are equal.

Product of a fuzzy set with a crisp number: Let A be a fuzzy set and α be a crisp value then the product operation results in a fuzzy set whose membership function is defined as follows $\mu_{\alpha.A}(x) = \alpha * \mu_A(x)$. Let α = 2, the operation is as follows

$\mu_{2.A}(x_1) = 2 * 0.2 = 0.4$

$\mu_{2.A}(x_2) = 2 * 0.3 = 0.6$

$\mu_{2.A}(x_3) = 2 * 0.7 = 1.4 = 1$

$\mu_{2.A}(x_4) = 2 * 0.5 = 1$

$2.A = \left\{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{1}{x_3}, \frac{0.5}{x_4}\right\}$

Product of two fuzzy set: Let A and B be two fuzzy sets then the product operation results in a fuzzy set whose membership function is defined as follows $\mu_{A.B}(x) = \mu_A(x).\mu_B(x)$.

$\mu_{A.B}(x_1) = 0.2 * 0.6 = 0.12,$

$\mu_{A.B}(x_2) = 0.3 * 0.1 = 0.03,$

$\mu_{A.B}(x_3) = 0.7 * 0.6 = 0.35,$

$\mu_{A.B}(x_4) = 0.5 * 0.4 = 0.2.$

$A.B = \left\{\frac{0.12}{x_1}, \frac{0.03}{x_2}, \frac{0.35}{x_3}, \frac{0.2}{x_4}\right\}$

Power of a fuzzy set: Let A be a fuzzy set and α be a crisp value then the power operation results in a fuzzy set whose membership function is defined as follows $\mu_{A^\alpha}(x) = \{\mu_A(x)\}^\alpha$. Let α = 2, the operation is as follows

$\mu_{A^2}(x_1) = \{0.2\}^2 = 0.04,$

$\mu_{A^2}(x_2) = \{0.3\}^2 = 0.09,$

$\mu_{A^2}(x_3) = \{0.7\}^2 = 0.49,$

$\mu_{A^2}(x_4) = \{0.5\}^2 = 0.25.$

$A^2 = \left\{\frac{0.04}{x_1}, \frac{0.09}{x_2}, \frac{0.49}{x_3}, \frac{0.25}{x_4}\right\}$

Difference: Let A and B be two fuzzy sets then the difference operation results in a fuzzy set whose membership function is defined in terms of intersection and complementation operation as follows $\mu_{A-B}(x) = \min\{\mu_A(x), 1 - \mu_B(x)\}$.

$\mu_{A-B}(x_1) = \min\{0.2, 1 - 0.6\} = \min(0.2, 0.4) = 0.2,$

$\mu_{A-B}(x_2) = \min\{0.3, 1 - 0.1\} = \min(0.3, 0.9) = 0.3,$

$\mu_{A-B}(x_3) = \min\{0.7, 1 - 0.6\} = \min(0.7, 0.4) = 0.4,$

$\mu_{A-B}(x_4) = \min\{0.5, 1 - 0.4\} = \min(0.5, 0.6) = 0.5.$

$A - B = \left\{\frac{0.2}{x_1}, \frac{0.3}{x_2}, \frac{0.4}{x_3}, \frac{0.5}{x_4}\right\}$

Disjunctive sum: Let A and B be two fuzzy sets then the disjunctive sum operation results in a fuzzy set whose membership function is defined as follows $A \oplus B = (A^c \cap B) \cup (A \cap B^c)$. And the membership function for the resultant set is as follows $\mu_{A\oplus B}(x) = \max\{\min\{1 - \mu_A(x), \mu_B(x)\}, \min\{\mu_A(x), 1 - \mu_B(x)\}\}$.

$\mu_{A\oplus B}(x) = \max\{\min\{1 - \mu_A(x), \mu_B(x)\}, \min\{\mu_A(x), 1 - \mu_B(x)\}\}$

$\mu_{A\oplus B}(x_1) = \max\{\min\{1 - 0.2, 0.6\}, \min\{0.2, 1 - 0.6\}\} = \max\{\min\{0.8, 0.6\}, \min\{0.2, 0.4\}\} = \max\{0.6, 0.2\} = 0.6,$

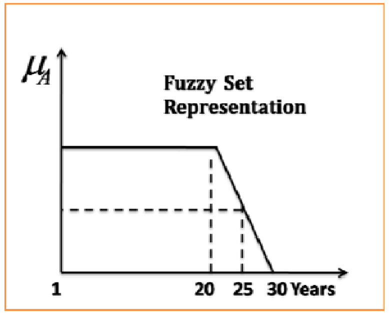
$\mu_{A\oplus B}(x_2) = \max\{\min\{1 - 0.3, 0.1\}, \min\{0.3, 1 - 0.1\}\} = \max\{\min\{0.7, 0.1\}, \min\{0.3, 0.9\}\} = \max\{0.1, 0.3\} = 0.3,$

$\mu_{A\oplus B}(x_3) = \max\{\min\{1 - 0.7, 0.6\}, \min\{0.7, 1 - 0.6\}\} = \max\{\min\{0.3, 0.6\}, \min\{0.7, 0.4\}\} = \max\{0.3, 0.4\} = 0.4,$

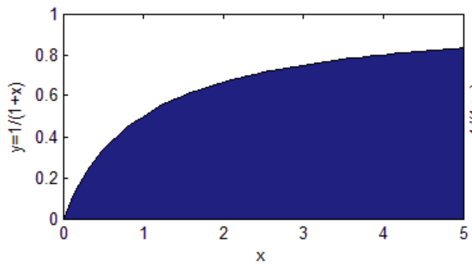
$\mu_{A\oplus B}(x_4) = \max\{\min\{1 - 0.5, 0.4\}, \min\{0.5, 1 - 0.4\}\} = \max\{\min\{0.5, 0.4\}, \min\{0.5, 0.6\}\} = \max\{0.5, 0.5\} = 0.5.$

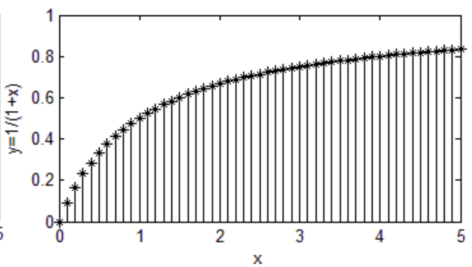
MEMBERSHIP FUNCTIONS

Fuzzyness in a fuzzy set is characterized by its membership function. Membership function is defined as a mapping from U to [0,1], where U is the universal set and the range [0,1] represents its membership value. membership function for a set A is represented by $\pi_A(x)$, where $x \in$U and $\pi_A(x) \in [0,1]$. Since $\pi_A(x)$ is a function of a single variable *x*, so we draw a 2D plot for $\pi_A(x)$ for all *x* in U.



Types of membership function: Membership function can be divided into two i.e. continuous and discrete.





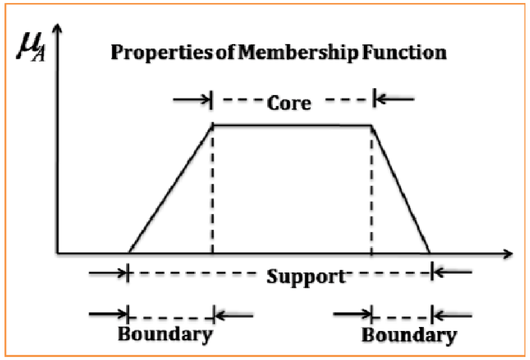
Features of membership function:

A membership function is characterised by three unique properties which one different form the other. They are as follows:

i. ***Core:*** It is that part of membership function where the membership value is 1 for some elements of the given fuzzy set. i.e. $\mu_A(x) = 1$.

ii. ***Support:*** It is that part of membership function where the membership value is non zero for some elements of the given fuzzy set. i.e. $\mu_A(x) \neq 0$.

iii. ***Boundary:*** It is that part of membership function where the membership value is in between 0 and 1 for some elements of the given fuzzy set. i.e. $0 < \mu_A(x) < 1$.

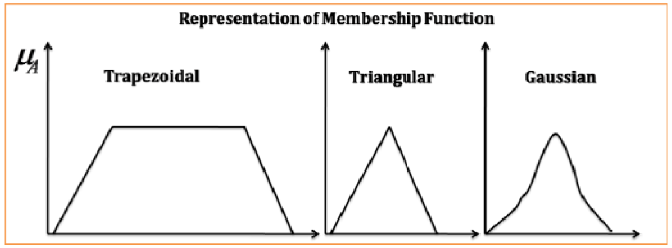


Cross over point: The elements of the fuzzy set of the points in the membership function where the membership value is 0.5 is called the crossover point. i.e. $\mu_A(x) = 0.5$. Basically there are two crossover point for any membership function.

Height: It the highest membership value for a membership function. i.e. $Height = \max\{\mu_A(x)\}$.

Representation of fuzzy sets:

Depending on situation the membership functions takes different shapes. The basic representations are *Trapezoidal, Triangular* and *Gaussian.*



Example:

$\mu_A(x) = \frac{x}{x+1}$; $\mu_B(x) = 2^{-x}$ *and* $x: [0 - 5]$. Determine the mathematical formulae and graphs for each of the following sets.

a. A^C, B^C

b. $A \cup B$

c. $A \cap B$

d. $(A \cup B)^c$

Solution:

a. $\mu_A(x) = \frac{x}{x+1}$; $x = [0 - 5]$

$\mu_{A^c}(x) = 1 - \frac{x}{x+1} = \frac{x+1-x}{x+1} = \frac{1}{x+1}$

$\mu_B(x) = 2^{-x}$; $x = [0 - 5]$

$\mu_{B^c}(x) = 1 - \frac{1}{2^x} = \frac{2^x - 1}{2^x}$

b. $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$; $x = [0 - 5]$

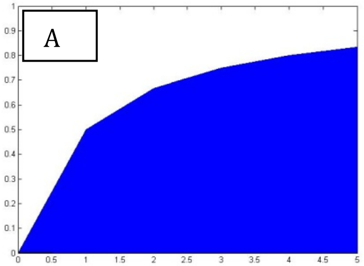
$\mu_{A\cup B}(x) = \max\left\{\frac{x}{x+1}, 2^{-x}\right\}$

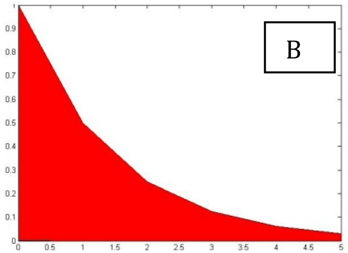
c. $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$; $x = [0 - 5]$

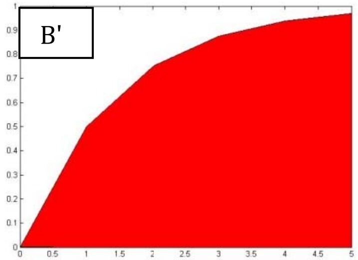
$\mu_{A\cap B}(x) = \min\left\{\frac{x}{x+1}, 2^{-x}\right\}$

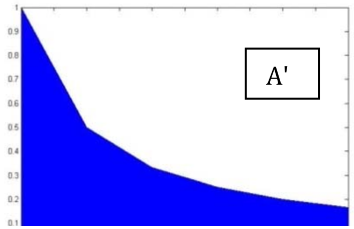
d. $\mu_{(A\cup B)^c}(x) = \min\{\mu_{A^c}(x), \mu_{B^c}(x)\}$; $x = [0 - 5]$

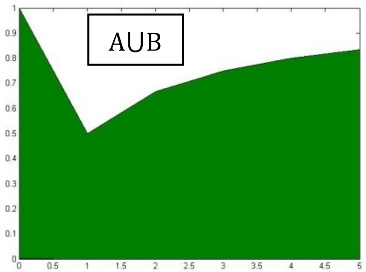
$\mu_{(A\cup B)^c}(x) = \min\left\{\frac{1}{x+1}, \frac{2^x - 1}{2^x}\right\}$

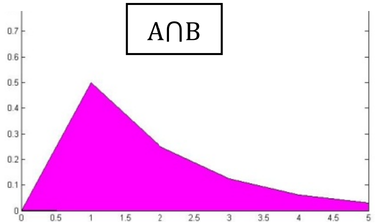


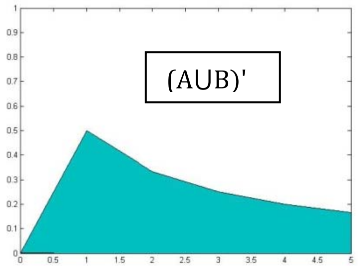












RELATIONS IN CRISP LOGIC

A relation R in crisp logic is defined as a subset of Cartesian product of two crisp set A and B i.e. $R \subseteq A \times B$. The elements of a relations are ordered pairs i.e. *(x_i, y_j)* such that x_i∈A and y_j∈B.

Let A={1, 2, 3} and B={2, 3, 4}, then

A×B={(1,2), (2,2), (3,2), (1,3), (2,3), (3,3), (1,4), (2,4), (3,4)} and

let *R={y=x+1|x ∈A and y∈B}*. Then R={(1,2),{2,3),(3,4)}.

R	1	2	3
2	1	0	0
3	0	1	0
4	0	0	1

OPERATIONS ON RELATIONS

Relations are nothing but sets intern, so the operations that can be applied to set can be also applied to relations also. Let R and S be the two relations.

R={y=x+1|x ∈A and y∈B}. Then R={(1,2),{2,3),(3,4)}.

S={y=x|x ∈A and y∈B}. Then s={{2,2),(3,3)}.

S	1	2	3
2	0	1	0
3	0	0	1
4	0	0	0

The operations on sets are as follows:

i. ***Union:*** The union operation on two relation R and S is defined as follows: $R \cup S(x,y) = \max\{R(x,y), S(x,y)\}$.

R	1	2	3	S	1	2	3	$R \cup S$	1	2	3
2	1	0	0	2	0	1	0	2	1	1	0
3	0	1	0	3	0	0	1	3	0	1	1
4	0	0	1	4	0	0	0	4	0	0	1

ii. ***Intersection:*** The intersection operation on two relation R and S is defined as follows: $R \cap S(x,y) = \min\{R(x,y), S(x,y)\}$.

R	1	2	3	S	1	2	3	$R \cap S$	1	2	3
2	1	0	0	2	0	1	0	2	0	0	0
3	0	1	0	3	0	0	1	3	0	0	0
4	0	0	1	4	0	0	0	4	0	0	0

iii. ***Complementation:*** The complementation operation on a relation R and S is defined as follows: $R^C(x,y) = 1 - R(x,y)$.

R	1	2	3	R'	1	2	3	S	1	2	3	S'	1	2	3
2	1	0	0	2	0	1	1	2	0	1	0	2	1	0	1
3	0	1	0	3	1	0	1	3	0	0	1	3	1	1	0
4	0	0	1	4	1	1	0	4	0	0	0	4	1	1	1

iv. ***Composition:*** The composition operation on two relation R and S is defined as follows: $RoS(x,z) = \max\{minR(x,y), S(y,z)\}$.

R	1	2
2	1	0
3	0	1
4	0	0

S	1	2	3
1	1	0	0
2	0	1	0

RoS	1	2	3
2	1	0	0
3	0	1	0
4	0	0	0

RELATIONS IN FUZZY LOGIC

A relation R in fuzzy logic is defined in a similar manner like that of crisp logic as a sunset of Cartesian product of two fuzzy set A and B i.e. $R \subseteq A \times B$. The elements of a fuzzy relations are represented as $\mu_{ARB}(x,y)$, where 'R' represent the relation. Let us take two sets: $A = \left\{\frac{0.2}{x_1}, \frac{0.7}{x_2}, \frac{0.4}{x_3}\right\}$ and $B = \left\{\frac{0.5}{y_1}, \frac{0.6}{y_2}\right\}$,

Then $A \times B$ is defined as a set whose membership function is given by the relation $\mu_{A \times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$.

$A \times B =$
$\{[(x_1,y_1),0.2],[(x_1,y_2),0.2],[(x_2,y_1),0.5],[(x_2,y_2),0.6],[(x_3,y_1),0.4],[(x_3,y_2),0.4]\}$

The same thing be represented in the form of a table as above.

AxB	y_1	y_2
x_1	0.2	0.2
x_2	0.5	0.6
x_3	0.4	0.4

OPERATIONS ON FUZZY RELATIONS

Similar to crisp logic we also have a similar set of operations on fuzzy logic. To understand the operations let us take two fuzzy relations R and S as follows.

Union, intersection requires two relations to union compatible i.e. R and S should be same order =mxn in both R and S.

R	y_1	y_2
x_1	0.7	0.6
x_2	0.8	0.3

S	y_1	y_2
x_1	0.8	0.5
x_2	0.1	0.6

The operations on fuzzy logic are as follows:

i. ***Union:*** The union operation on two fuzzy relation R and S is defined as follows: $\mu_{R \cup S}(x,y) = \max\{\mu_R(x,y), \mu_S(x,y)\}$.

$\mu_{R \cup S}(x_1,y_1) = \max\{\mu_R(x_1,y_1), \mu_S(x_1,y_1)\}$ =max{0.7,0.8} =0.8,
$\mu_{R \cup S}(x_1,y_2) = \max\{\mu_R(x_1,y_2), \mu_S(x_1,y_2)\}$ =max{0.6,0.5} =0.6,
$\mu_{R \cup S}(x_2,y_1) = \max\{\mu_R(x_2,y_1), \mu_S(x_2,y_1)\}$ =max{0.8,0.1} =0.8,
$\mu_{R \cup S}(x_2,y_2) = \max\{\mu_R(x_2,y_2), \mu_S(x_2,y_2)\}$ =max{0.3,0.6} =0.6.

RUS	y_1	y_2
x_1	0.8	0.6
x_2	0.8	0.6

ii. ***Intersection:*** The intersection operation on two fuzzy relation R and S is defined as follows: $\mu_{R \cap S}(x,y) = \min\{\mu_R(x,y), \mu_S(x,y)\}$.

$\mu_{R \cap S}(x_1,y_1) = \min\{\mu_R(x_1,y_1), \mu_S(x_1,y_1)\}$ =min{0.7,0.8} =0.7,
$\mu_{R \cap S}(x_1,y_2) = \min\{\mu_R(x_1,y_2), \mu_S(x_1,y_2)\}$ =min{0.6,0.5} =0.5,
$\mu_{R \cap S}(x_2,y_1) = \min\{\mu_R(x_2,y_1), \mu_S(x_2,y_1)\}$ =min{0.8,0.1} =0.1,
$\mu_{R \cap S}(x_2,y_2) = \min\{\mu_R(x_2,y_2), \mu_S(x_2,y_2)\}$ =min{0.3,0.6} =0.3.

RnS	y_1	y_2
x_1	0.7	0.5
x_2	0.1	0.3

iii. ***Complementation:*** The complementation operation on a fuzzy relation R is defined as follows: $\mu_{R^c}(x,y) = 1 - \mu_R(x,y)$.

$\mu_{R^c}(x_1,y_1) = 1 - \mu_R(x_1,y_1)$ =1-0.7 =0.3,

$\mu_{R^c}(x_1,y_2) = 1 - \mu_R(x_1,y_2)$ =1-0.6 =0.4,

$\mu_{R^c}(x_2,y_1) = 1 - \mu_R(x_2,y_1)$ =1-0.8 =0.2,

$\mu_{R^c}(x_2,y_2) = 1 - \mu_R(x_2,y_2)$ =1- 0.3=0.7.

R^c	y_1	y_2
x_1	0.3	0.4
x_2	0.2	0.7

iv. ***Composition:*** For composition operation R and S may not be necessary but they have number of columns of first relation equals to numbers of rows of second relations. i.e. if order of R(x, y) and S(y, z) are m x n and n x p respectively, the composition is possible and the resultant relation is of order m x p. Let the R and S be represented as follows:

R	y_1	y_2	y_3
x_1	0.7	0.6	0.2
x_2	0.8	0.3	0.4

S	z_1	z_2
y_1	0.8	0.5
y_2	0.1	0.6
y_2	0.7	0.2

a. Max-Min: The Max-Min composition on two fuzzy relation R and S is defined as follows: $\mu_{RoS}(x,z) = \max\{\min\{\mu_R(x,y), \mu_S(y,z)\}\} | x \in X, y \in Y, z \in Z$.

$$\mu_{RoS}(x_1,z_1) = \max\left\{\begin{matrix}\min\{\mu_R(x_1,y_1),\mu_S(y_1,z_1)\},\\ \min\{\mu_R(x_1,y_2),\mu_S(y_2,z_1)\},\\ \min\{\mu_R(x_1,y_3),\mu_S(y_2,z_1)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\min\{0.7,0.8\},\\ \min\{0.6,0.1\},\\ \min\{0.2,0.7\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.7\\0.1\\0.2\end{matrix}\right\} = 0.7$$

$$\mu_{RoS}(x_1,z_2) = \max\left\{\begin{matrix}\min\{\mu_R(x_1,y_1),\mu_S(y_1,z_2)\},\\ \min\{\mu_R(x_1,y_2),\mu_S(y_2,z_2)\},\\ \min\{\mu_R(x_1,y_3),\mu_S(y_2,z_2)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\min\{0.7,0.5\},\\ \min\{0.6,0.6\},\\ \min\{0.2,0.2\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.5\\0.6\\0.2\end{matrix}\right\} = 0.5$$

$$\mu_{RoS}(x_2,z_1) = \max\left\{\begin{matrix}\min\{\mu_R(x_2,y_1),\mu_S(y_1,z_1)\},\\ \min\{\mu_R(x_2,y_2),\mu_S(y_2,z_1)\},\\ \min\{\mu_R(x_2,y_3),\mu_S(y_2,z_1)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\min\{0.8,0.8\},\\ \min\{0.3,0.1\},\\ \min\{0.4,0.7\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.8\\0.1\\0.4\end{matrix}\right\} = 0.8$$

$$\mu_{RoS}(x_2,z_2) = \max\left\{\begin{matrix}\min\{\mu_R(x_2,y_1),\mu_S(y_1,z_2)\},\\ \min\{\mu_R(x_2,y_2),\mu_S(y_2,z_2)\},\\ \min\{\mu_R(x_2,y_3),\mu_S(y_2,z_2)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\min\{0.8,0.5\},\\ \min\{0.3,0.6\},\\ \min\{0.4,0.2\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.5\\0.3\\0.2\end{matrix}\right\} = 0.5$$

RoS	z_1	z_2
x_1	0.7	0.5
x_2	0.8	0.5

b. Max-Product: The Max-Product composition on two fuzzy relation R and S is defined as follows: $\mu_{RoS}(x,z) = \max\{\mu_R(x,y) * \mu_S(y,z)\} | x \in X, y \in Y, z \in Z$.

$$\mu_{RoS}(x_1,z_1) = \max\left\{\begin{matrix}\{\mu_R(x_1,y_1) * \mu_S(y_1,z_1)\},\\ \{\mu_R(x_1,y_2) * \mu_S(y_2,z_1)\},\\ \{\mu_R(x_1,y_3) * \mu_S(y_2,z_1)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.7 * 0.8\},\\ \{0.6 * 0.1\},\\ \{0.2 * 0.7\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.56\\0.06\\0.14\end{matrix}\right\} = 0.56$$

$$\mu_{RoS}(x_1,z_2) = \max\left\{\begin{matrix}\{\mu_R(x_1,y_1) * \mu_S(y_1,z_2)\},\\ \{\mu_R(x_1,y_2) * \mu_S(y_2,z_2)\},\\ \{\mu_R(x_1,y_3) * \mu_S(y_2,z_2)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.7 * 0.5\},\\ \{0.6 * 0.6\},\\ \{0.2 * 0.2\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.35\\0.36\\0.04\end{matrix}\right\} = 0.36$$

$$\mu_{RoS}(x_2,z_1) = \max\left\{\begin{matrix}\{\mu_R(x_2,y_1) * \mu_S(y_1,z_1)\},\\ \{\mu_R(x_2,y_2) * \mu_S(y_2,z_1)\},\\ \{\mu_R(x_2,y_3) * \mu_S(y_2,z_1)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.8 * 0.8\},\\ \{0.3 * 0.1\},\\ \{0.4 * 0.7\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.64\\0.03\\0.28\end{matrix}\right\} = 0.64$$

RoS	z_1	z_2
x_1	0.56	0.36
x_2	0.64	0.40

$$\mu_{RoS}(x_2,z_2) = \max\left\{\begin{matrix}\{\mu_R(x_2,y_1) * \mu_S(y_1,z_2)\},\\ \{\mu_R(x_2,y_2) * \mu_S(y_2,z_2)\},\\ \{\mu_R(x_2,y_3) * \mu_S(y_2,z_2)\}.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.8 * 0.5\},\\ \{0.3 * 0.6\},\\ \{0.4 * 0.2\}.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.40\\0.18\\0.08\end{matrix}\right\} = 0.40$$

c. ***Max-Average:*** The Max-Average composition on two fuzzy relation R and S is defined as follows: $\mu_{RoS}(x,z) = \max\{(\mu_R(x,y) + \mu_S(y,z))/2 | x \in$ *X, y∈Y, z∈Z.*

$$\mu_{RoS}(x_1,z_1)$$

$$= \max\left\{\begin{matrix}\{\mu_R(x_1,y_1) + \mu_S(y_1,z_1)\}/2,\\ \{\mu_R(x_1,y_2) + \mu_S(y_2,z_1)\}/2,\\ \{\mu_R(x_1,y_3) + \mu_S(y_2,z_1)\}/2.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.7 + 0.8\}/2,\\ \{0.6 + 0.1\}/2,\\ \{0.2 + 0.7\}/2.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.75\\0.35\\0.45\end{matrix}\right\} = 0.75$$

$$\mu_{RoS}(x_1,z_2)$$

$$= \max\left\{\begin{matrix}\{\mu_R(x_1,y_1) + \mu_S(y_1,z_2)\}/2,\\ \{\mu_R(x_1,y_2) + \mu_S(y_2,z_2)\}/2,\\ \{\mu_R(x_1,y_3) + \mu_S(y_2,z_2)\}/2.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.7 + 0.5\}/2,\\ \{0.6 + 0.6\}/2,\\ \{0.2 + 0.2\}/2.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.60\\0.60\\0.20\end{matrix}\right\} = 0.60$$

$$\mu_{RoS}(x_2,z_1)$$

$$= \max\left\{\begin{matrix}\{\mu_R(x_2,y_1) + \mu_S(y_1,z_1)\}/2,\\ \{\mu_R(x_2,y_2) + \mu_S(y_2,z_1)\}/2,\\ \{\mu_R(x_2,y_3) + \mu_S(y_2,z_1)\}/2.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.8 + 0.8\}/2,\\ \{0.3 + 0.1\}/2,\\ \{0.4 + 0.7\}/2.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.80\\0.20\\0.55\end{matrix}\right\} = 0.80$$

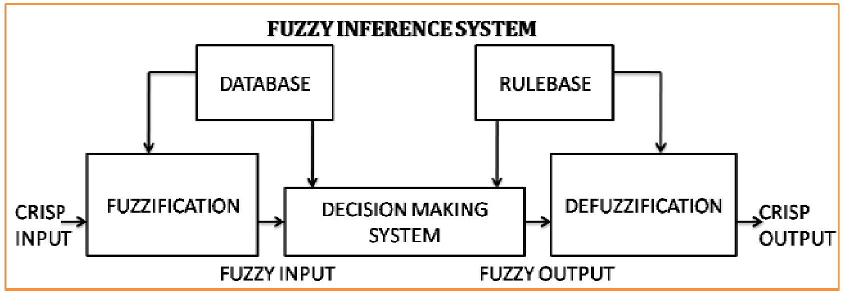
$$\mu_{RoS}(x_2,z_2)$$

$$= \max\left\{\begin{matrix}\{\mu_R(x_2,y_1) + \mu_S(y_1,z_2)\}/2,\\ \{\mu_R(x_2,y_2) + \mu_S(y_2,z_2)\}/2,\\ \{\mu_R(x_2,y_3) + \mu_S(y_2,z_2)\}/2.\end{matrix}\right\} = max\left\{\begin{matrix}\{0.8 + 0.5\}/2,\\ \{0.3 + 0.6\}/2,\\ \{0.4 + 0.2\}/2.\end{matrix}\right\}$$

$$= max\left\{\begin{matrix}0.65\\0.45\\0.30\end{matrix}\right\} = 0.65$$

RoS	z_1	z_2
x_1	0.75	0.60
x_2	0.80	0.65

FIS: FUZZY INFERENCE SYSTEM



The fuzzy inference system basically consists of five functional units as follows:

i. ***Database:*** The linguistic variable are represented as membership function and this is the place where the membership functions are stored.

ii. ***Rulebase:*** The linguistic variables are combined with different connectives and inferences to form rules and this is where these rules are stored in the form of *"if then"* rules.

iii. ***Decision making system:*** It receives the fuzzy inputs from the fuzzification unit and processes it with the relations stored in the system to give fuzzy output to the defuzzification unit.

iv. ***Fuzzification unit:*** It is the unit, which actually receives the input from the real world that are crisp in nature. It the converts them into fuzzy inputs by different fuzzification techniques and forwards it to the decision making system.

v. ***Defuzzification unit:*** It is the unit, which is responsible for giving output to the real world that, are crisp in nature. It receives the fuzzy output form the decision making system and applies different defuzzification techniques to convert the same into crisp output.

CRISP LOGIC OR BOOLEAN LOGIC OR CLASSICAL LOGIC

In crisp logic we represent the fact of real world interms of proposition, connectives and inference rules.

Proposition: A fact in the real world is represented as a proposition. i.e. *"Ram is a good boy"* is represented as P(x) where x represents Ram. Proposition of these kinds are called as atomic or simple proposition. We can obtain complex propositions by making use of connectives.

Connectives: These are used to build complex propositions with the help of atomic propositions. The connectives are as follows:

P	Q	~P	~Q	P∨Q	P ∧ Q	P ⟹ Q	~P∨Q
T	T	F	F	T	T	T	T
T	F	F	T	T	F	F	F
F	T	T	F	T	F	T	T
F	F	T	T	F	F	T	T

Inference Rules: By making use of given propositions we can infer some unknown fact. This mechanism is called as *inferenceing.* There are two kinds of inference rules that are as follows:

i. ***Modus Ponens:*** It states that if P is true and P=>Q is true, then we infer Q is true. Here we infer P as a fact and P=>Q is a rule and Q as inference.

P	*is true*	*is a fact*
P=>Q	*is true*	*is a rule*
Q	*is true*	*is inferred.*

ii. ***Modus Tollens:*** It states that if ~Q is true and P=>Q is true, then we infer ~P is true. Here we infer ~Q as a fact and P=>Q is a rule and ~P as inference.

~Q	*is true*	*is a fact*
P=>Q	*is true*	*is a rule*
~P	*is true*	*is inferred.*

FUZZY LOGIC

In crisp logic we consider a proposition to be either true or false i.e. T/F, but in fuzzy logic we consider a proposition to take fuzzy truth-values i.e. values between 0 and 1.

Fuzzy Proposition: A fuzzy proposition is represented as T(P) which takes truth values between 0 and 1. A fuzzy proposition similar to fuzzy sets are represented by a membership functions i.e. $T(\tilde{A}) = \mu_{\tilde{A}}(x)$ *such that* $0 \leq \mu_{\tilde{A}}(x) \leq 1$. E.g. Let $\tilde{A}$=Ram is good, then $T(\tilde{A})$=0.8 implies the statement is partially true, where as $T(\tilde{A})$=1 implies the statement is absolutely true.

Connectives: Complex or compound fuzzy propositions are constructed by making use of connectives. The use of connectives in fuzzy logic is as follows:

Connectives	*Membership Functions*
~P	1-T(P)
P∨Q	*max(T(P),T(Q))*
P ∧ Q	*min(T(P),T(Q))*
P ⟹ Q or *~P∨Q*	*max(1-T(P),T(Q))*

Inference Procedure: In fuzzy logic we represent the rules in terms of *"if then"* rules. i.e. if we have two fuzzy proposition A and B and we have A=>B, then this is interpreted as *"if x is in A then y is in B"*. Another form rule i.e. *"if x in A then y in B else y in C"* is also possible. We represent these rules in terms of a fuzzy relation R, that takes different form depending on the rule for which is designed i.e. if we are making use of *"If x in A then y in B"* then $R = (A \times B) \cup (A^C \times Y)$, and *"if x in A then y in B else y in C"* then $R = (A \times B) \cup (A^C \times C)$.

Example:
Let $X = \{a, b, c, d\}$, *and* $Y = \{1, 2, 3, 4\}$ be two sets.
Let A be a fuzzy proposition defined on the set X as $\tilde{A} = \left\{\frac{0.0}{a}, \frac{0.8}{b}, \frac{0.6}{c}, \frac{1.0}{d}\right\}$,
Let B, C be fuzzy propositions defined on the set Y as $\tilde{B} = \left\{\frac{0.2}{1}, \frac{1.0}{2}, \frac{0.8}{3}, \frac{0.0}{4}\right\}$ and $\tilde{C} = \left\{\frac{0.0}{1}, \frac{0.4}{2}, \frac{1.0}{3}, \frac{0.8}{4}\right\}$ respectively.
Then for the rule *"if x in A then y in B"*, we have $R = (\tilde{A} \times \tilde{B}) \cup (\widetilde{A^C} \times \tilde{Y})$, where $\tilde{Y}$ is the fuzzy set representing the set Y such that each element has membership function equal to 1, i.e. $\tilde{Y} = \left\{\frac{1.0}{1}, \frac{1.0}{2}, \frac{1.0}{3}, \frac{1.0}{4}\right\}$.

So $\mu_R(x, y) = \max\{\min\{\mu_A(x), \mu_B(y)\}, \min\{1 - \mu_A(x), \mu_Y(y)\}\}$, this can be rewritten as $\mu_R(x, y) = \max\{\min\{\mu_A(x), \mu_B(y)\}, 1 - \mu_A(x)\}$ as $\mu_Y(y)$ is always 1 and will be greater than 1-$\mu_A(x)$ so $\min\{1 - \mu_A(x), \mu_Y(y)\}$ is equivalent to 1-$\mu_A(x)$.

AxB	1	2	3	4
a	0.0	0.0	0.0	0.0
b	0.2	0.8	0.8	0.0
c	0.2	0.6	0.6	0.0
d	0.2	1.0	0.8	0.0

A^cxY	1	2	3	4
A	1.0	1.0	1.0	1.0
B	0.2	0.2	0.2	0.2
C	0.4	0.4	0.4	0.4
D	0.0	0.0	0.0	0.0

R	1	2	3	4
a	1.0	1.0	1.0	1.0
b	0.2	0.8	0.8	0.2
c	0.4	0.6	0.6	0.4
d	0.2	1.0	0.8	0.0

For the rule *"if x in A then y in B else y in C"*, we have $R = (\tilde{A} \times \tilde{B}) \cup (\widetilde{A^C} \times \tilde{C})$.

So $\mu_R(x, y) = \max\{\min\{\mu_A(x), \mu_B(y)\}, \min\{1 - \mu_A(x), \mu_C(y)\}\}$.

$\widetilde{A^C} = \left\{\frac{1.0}{a}, \frac{0.2}{b}, \frac{0.4}{c}, \frac{0.0}{d}\right\}, \tilde{C} = \left\{\frac{0.0}{1}, \frac{0.4}{2}, \frac{1.0}{3}, \frac{0.8}{4}\right\}$

AxB	1	2	3	4
a	0.0	0.0	0.0	0.0
b	0.2	0.8	0.8	0.0
c	0.2	0.6	0.6	0.0
d	0.2	1.0	0.8	0.0

A^cxC	1	2	3	4
a	0.0	0.4	1.0	0.8
b	0.2	0.4	0.2	0.2
c	0.0	0.4	0.4	0.4
d	0.0	0.0	0.0	0.0

R	1	2	3	4
a	0.0	0.4	1.0	0.8
b	0.2	0.8	0.8	0.2
c	0.2	0.6	0.6	0.4
d	0.2	1.0	0.8	0.0

In Fuzzy we have two inference processes i.e. Generalized Modus ponens (GMP) and Generalized Modus Tollens (GMT). The inference process in fuzzy logic is as follows:

i. ***Generalized Modus Ponens (GMP):*** It works when a fuzzy proposition P and a rule P=>Q is given and we need to infer Q.

Example: *P="x in A"* *Given fact*
if "x in A" then "y in B" *Given rule*
Q="y in B" *Inference*

This can be also shown by composition operation also i.e. $\tilde{B} = \tilde{A}\ o\ \tilde{R}$ where $R = (\tilde{A} \times \tilde{B}) \cup (\widetilde{A^C} \times \tilde{Y})$.

Example:
GMP can also be applied to problem where we have more than one variable i.e. *"if old and car is high power then risk is high"*. So here GMP take the form *"if x in A and y in B then z in C"*

P="x in A" and Q="y in B" *Given fact*
if "x in A" and "y in B" then "z in C" *Given rule*
S="z in C" *Inference*

Here we make use of $\tilde{C} = (\tilde{A} \cup \tilde{B})\, o\, \tilde{R}$

ii. ***Generalized Modus Tollens (GMT):*** It works when a fuzzy proposition ~Q and a rule P=>Q is given and we need to infer ~P.

Example: *~Q="x not in A"* *Given fact*
if "x in A" then "y in B" *Given rule*
~P="y not in B" *Inference*

This can be also shown by composition operation also i.e. $\sim\tilde{A} = \sim\tilde{B}\ o\ \tilde{R}$ where $R = (\tilde{A} \times \tilde{B}) \cup (\widetilde{A^C} \times \tilde{Y})$.

Example:
GMP can also be applied to problem where we have more than one variable i.e. *"if risk not high then not old or car is not high power "*. So here GMP take the form *"if x in A and y in B then z in C"*

S="z not in C" *Given fact*
if "x in A" and "y in B" then "z in C" *Given rule*
P="x not in A" or Q="y not in B" *Inference*

Here we make use of $(\sim\tilde{A} \cap \sim\tilde{B}) = \sim\tilde{C}\, o\, \tilde{R}$

FUZZY UNION AND INTERSECTION

The intersection and union of two fuzzy sets can also be performed with T-norm and T-conorm or S-norm respectively. Let us discuss these operations in detail.

T-NORMS

T-norm operator is a two place function i.e. T(. , .) satisfying the following properties:

Properties:

i. ***Boundary:*** *T(0,0)=0, T(a,1)=T(1,a)=a* i.e. it imposes correct generalization to crisp sets.
ii. ***Monotonicity:*** *T(a, b)<=T(c, d) if a<c and b<d* i.e. increase in values of a and b also increased the value in *T(a, b)*.
iii. ***Commutative:*** *T(a, b)=T(b, a)* i.e. the operator is indifferent to the order of fuzzy sets to be combined.
iv. ***Associativity:*** *T(a, T(b, c))=T(T(a, b), c)* i.e. any number of fuzzy sets and in any order can be combined to form pair-wise grouping since T-norm is a two place function

Operation using T-norms:

The four operations that are allowed in T-norms are as follows:

i. ***Minimum:*** $T_{min}(a, b) = \min\{a, b\} = a \wedge b$.
ii. ***Algebraic product:*** $T_{ap}(a, b) = ab$.
iii. ***Bounded product:*** $T_{bp}(a, b) = 0 \vee (a + b - 1)$.
iv. ***Drastic product:*** $T_{dp}(a, b) = \begin{cases} a, if\ b = 1. \\ b, if\ a = 1. \\ 0, if\ a, b < 1. \end{cases}$

Relationship between the different operations in T-Norms

$$T_{dp}(a, b) \leq T_{bp}(a, b) \leq T_{ap}(a, b) \leq T_{min}(a, b)$$

S-NORM

S-norm operator also called as T-conorm is a two place function i.e. S(., .) satisfying the following properties:

Properties:

i. ***Boundary:*** *S(1,1)=1, S(a,0)=S(0,a)=a* i.e. it imposes correct generalization to crisp sets.
ii. ***Monotonicity:*** *S(a, b)<=S(c, d) if a<c and b<d* i.e. increase in values of a and b also increased the value in *T(a, b)*.
iii. ***Commutative:*** *S(a, b)=S(b, a)* i.e. the operator is indifferent to the order of fuzzy sets to be combined.
iv. ***Associativity:*** *S(a, S(b, c))=S(S(a, b), c)* i.e. any number of fuzzy sets and in any order can be combined to form pair-wise grouping since T-norm is a two place function

Operation using S-norms:

The four operations that are allowed in T-norms are as follows:

i. ***Maximum:*** $S_{max}(a,b) = \max\{a,b\} = a \vee b.$
ii. ***Algebraic sum:*** $S_{as}(a,b) = a + b - ab.$
iii. ***Bounded sum:*** $T_{bs}(a,b) = 1 \wedge (\mathrm{a} + \mathrm{b}).$
iv. ***Drastic sum:*** $T_{ds}(a,b) = \begin{cases} a, if\ b = 0. \\ b, if\ a = 0. \\ 1, if\ a,b > 0. \end{cases}$

Relationship between the different operations in T-Norms

$$S_{max}(a,b) \leq S_{ap}(a,b) \leq S_{bp}(a,b) \leq S_{ds}(a,b)$$

DECOMPOSITION

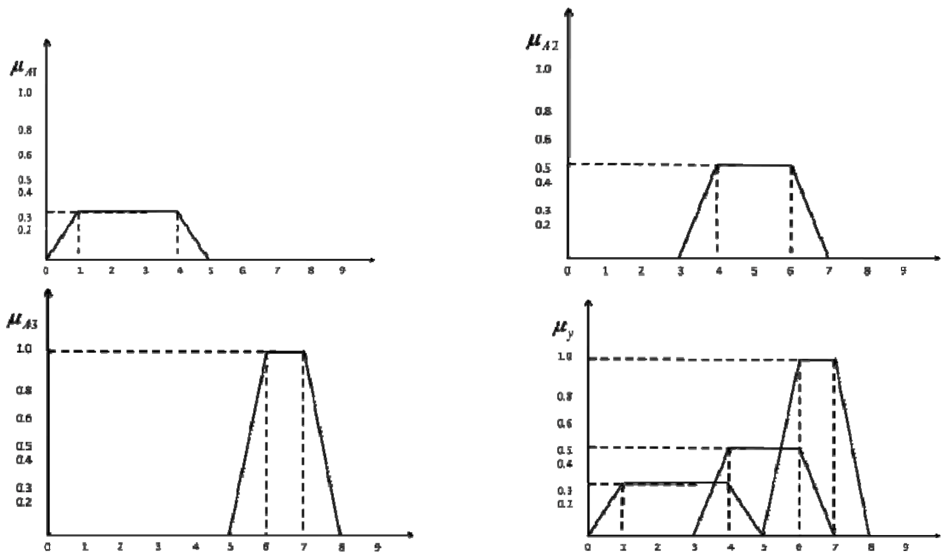
The output of decision-making system on application of fuzzy inference rule gives fuzzy outputs. These fuzzy outputs cannot be interpreted by the real world, so a conversion of fuzzy output to crisp output is highly necessary. This conversion is brought about with the *defuzzification process*. It acts as the last unit of fuzzy inference system this is responsible for giving crisp output to the users. The process is the reverse of *fuzzification process*. There are different method to carry out the defuzzification process which are as follows:

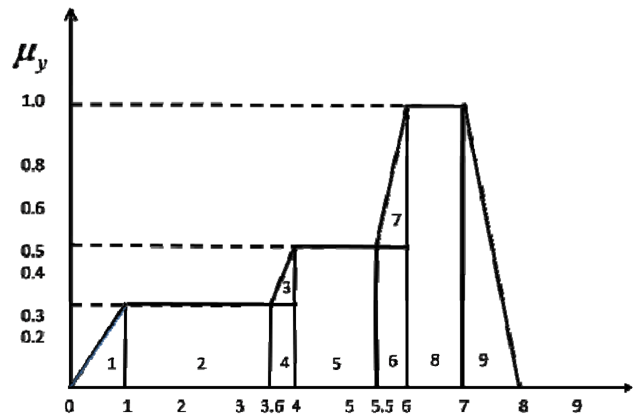
i. ***Centroid Method:*** From the given membership functions an envelope function is created that contain all the functions to scale. The envelope membership function is then divided into triangles and rectangle, each being called as a segment. The segments are numbered and for each of the segment the area and its centroid w.r.t the origin is calculated.

Segment	*Area*	$\bar{x}$	*Area* . $\bar{x}$
1	$\frac{1}{2}$ x 1 x 0.3 = 0.15	0.67	0.1005
2	2.6 x 0.3 = 0.78	$\frac{3.6-1}{2} + 1 = 2.3$	1.794
3	$\frac{1}{2}$ x 0.4 x 0.2 = 0.04	$3.6 + \frac{2}{3}$ x 0.4 = 3.8	0.154
4	0.4 x 0.3 = 0.12	$\frac{0.4}{2} + 3.6 = 3.8$	0.456
5	1.5 x 0.5 = 0.75	$\frac{1.5}{2} + 4 = 4.75$	3.5625
6	0.5 x 0.5 = 0.25	$\frac{0.5}{2} + 5.5 = 5.75$	1.4375
7	$\frac{1}{2}$ x 0.5 x 0.5 = 0.125	$\frac{1}{3}$ x 0.5 + 5.5 = 5.66	0.7075
8	1x 1 = 1	$\frac{1}{2} + 6 = 6.5$	6.5
9	$\frac{1}{2}$ x 1x 1 = $\frac{1}{2}$	$\frac{1}{3}$ x 1 + 7 = 7.3	3.665

After the all the areas and centroid is calculated, the centroid of the envelope is calculated by the use of the formulae as follows:

$$Centroid\ x^* = \frac{\sum A.\bar{x}}{\sum A} = \frac{18.433}{3.715} = 4.9617$$





ii. ***Centre of sum:*** In this method area and its centre form the origin is calculated for all the membership functions as follows:

Membership function	*Area*	*Centre from the origin* $\bar{x}$	*Area* . $\bar{x}$
1	$\frac{1}{2}$ x (3 + 5) x 0.3 = 1.2	2.5	3.0
2	$\frac{1}{2}$ x (4 + 2) x 0.5 = 1.5	5.0	7.5
3	$\frac{1}{2}$ x (3 + 1) x 1.0 = 1.2	6.5	13

After the all the areas and centre from the origin is calculated, the centre of sum is calculated by the use of the formulae as follows:

$$Centre\ of\ sum\ x^* = \frac{\sum A.\bar{x}}{\sum A} = \frac{23.5}{4.7} = 5.0$$

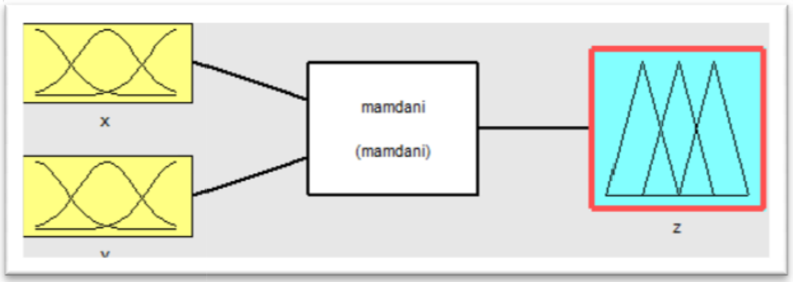
iii. ***Mean of maxima:*** In this method the maxima are found out from the envelope membership function and their mean is found out to get the centre as follows:

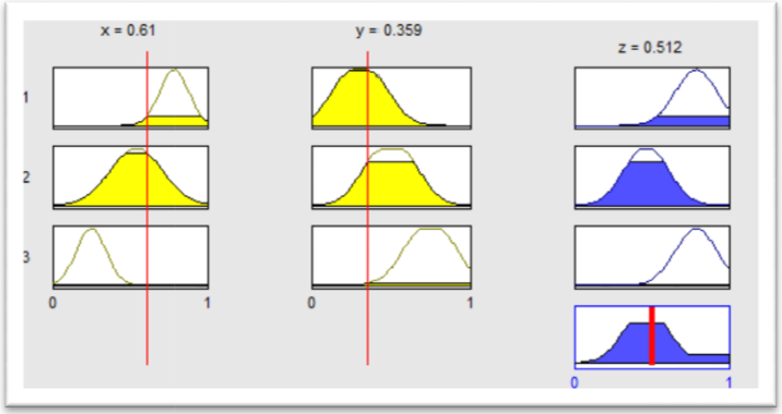
$$Centre\ \ x^* = \frac{\sum Maxima}{\sum no.of\ Maxima} = \frac{6+7}{2} = 6.5$$

CLASSIFICATION OF FUZZY INFERENCE SYSTEMS

There are two kinds of fuzzy inference system, they are as follows:

i. ***Mamdani FIS:*** In case of Mamdani fuzzy inference system, the input as well as the outputs are fuzzy sets. These fuzzy outputs are then defuzzified to give the crisp output.





ii. ***Takagi Sugeno and Kang FIS (TSK FIS):*** In case of TSK fuzzy inference system, the inputs are only fuzzy in nature and the outputs are not fuzzy, rather than they are functions of input variable i.e. *z(x, y)* where 'x' and 'y' are the input variables and 'z' is a function of x and y. The output function z may be a constant or linear.